

# Alpha Representation For Active Portfolio Management and High Frequency Trading In Seemingly Efficient Markets

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## Abstract

We introduce a trade strategy representation theorem for performance measurement and portable alpha in high frequency trading, by embedding a robust trading algorithm that describe portfolio manager market timing behavior, in a canonical multifactor asset pricing model. First, we present a spectral test for market timing based on behavioral transformation of the hedge factors design matrix. Second, we find that the typical trade strategy process is a local martingale with a background driving Brownian bridge that mimics portfolio manager price reversal strategies. Third, we show that equilibrium asset pricing models like the CAPM exists on a set with P-measure zero. So that excess returns, i.e. positive alpha, relative to a benchmark index is robust to no arbitrage pricing in turbulent capital markets. Fourth, the path properties of alpha are such that it is positive between suitably chosen stopping times for trading. Fifth, we demonstrate how, and why, econometric tests of portfolio performance tend to under report positive alpha.

*Keywords:* market timing; empirical alpha process; unobserved portfolio strategies; martingale system; behavioural finance; high frequency trading; Brownian bridge; Jensen's alpha; portable alpha

*JEL Classification Codes:* C02, G12, G13

## 1. Introduction

The problem posed is one in which a portfolio manager ("PM") wants to increase portfolio alpha—the returns on her portfolio, over and above a benchmark or market portfolio. To do so [s]he alters the betas<sup>1</sup> of the portfolio in anticipation of market movements by augmenting a benchmark model with hedge factors<sup>2</sup>—which includes but is not limited to revising asset allocation or readjusting portfolio weights within an asset class. In other words, altered betas represent the managers dynamic trading strategy<sup>3</sup>. Conceptually, the allocation of assets in the benchmark is "fixed" but hedge factors are stochastic<sup>4</sup>—at least for so called "portable alpha"<sup>5</sup>.

This paper's contribution to behavioural finance, and the gargantuan market timing literature, stems from its reconciliation of active portfolio management with efficient markets when portfolio strategy or investment style is unobservable<sup>6</sup>. It employs asymptotic theory to identify an empirical portfolio alpha process with

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<sup>1</sup>See e.g., [Grundy and Malkiel \(1996\)](#) for viability of beta as a useful metric for covariance with benchmarks. [Grinold \(1993\)](#) provides excellent exposition on the versatility of beta separate from its use in the CAPM introduced by [Sharpe \(1964\)](#), *inter alios*.

<sup>2</sup>See [Fama and French \(1996\)](#); and [\(Fung and Hsieh, 1997, pg. 276\)](#)

<sup>3</sup>Implicit in this assessment is the portfolio manager's response to good news or bad news accordingly—about assets in her portfolio—to exploit a so called leverage effect. See e.g., [Black \(1976\)](#); [Braun et al. \(1995\)](#).

<sup>4</sup>See e.g., [\(Jensen, 1967, pg. 10\)](#).

<sup>5</sup>See [\(Kung and Pohlman, 2004, pg. 78-79\)](#). To wit, the portfolio may be "market neutral" since benchmark and or market risk is hedged away..

<sup>6</sup>See e.g., [Henriksson and Merton \(1981\)](#); [Grinblatt and Titman \(1989\)](#); [Ferson and Schadt \(1996\)](#); [Mamaysky et al. \(2008\)](#); [Kacperczyk et al. \(2008\)](#).

dynamic portfolio adjustments<sup>7</sup> that reflect managerial strategy via martingale system equations that portend algorithmic trading. Additionally, it proves that the measurable sets for portfolio manager market timing ability are much larger than those proffered in the extant literature which tests for timing ability via statistical significance of convex payoff structure(s)<sup>8</sup>. Accordingly, we propose a new and simple test for market timing ability based on the spectral circle induced by a behavioural transformation of the hedge factor matrix.

The paper proceeds as follows. In [section 2](#) we formally introduce our model. Whereupon we summarize our representation theory result in Theorem [2.16](#). Our spectral test for market timing is presented in Proposition [2.13](#). In [section 3](#) we apply our theory to the ubiquitous CAPM to provide analytics about Jensen's alpha. The main result there is Theorem [3.1](#) on the path process of positive alpha.

## 2. The Canonical Linear Asset Pricing Model

Let

$$\mathbf{y} = X\boldsymbol{\delta} + Z\boldsymbol{\gamma} + \boldsymbol{\varepsilon} \quad (2.1)$$

be the canonical hedge factor model, i.e., augmented capital asset pricing model (CAPM), for a portfolio comprised of:  $X$ —a matrix of returns from *benchmark assets*<sup>9</sup>; and  $Z$ —a matrix of returns from *hedge factors*<sup>10</sup> mimicking derivatives. The portfolio *beta* is given by the row vector  $\boldsymbol{\beta}^T = (\boldsymbol{\delta}^T \ \boldsymbol{\gamma}^T)$  and  $\boldsymbol{\varepsilon}$  is a column vector of idiosyncratic error terms<sup>11</sup>. The hedge factor strategy is embodied by  $Z$ . Thus, *modulo* idiosyncratic error, our *portfolio alpha* is given by

$$\boldsymbol{\alpha} = Z\boldsymbol{\gamma} \quad (2.2)$$

Whereupon  $\boldsymbol{\gamma}$  is hedge factor exposure sensitivity—it represents the *trading strategy* of the portfolio manager<sup>12</sup>. Similarly,  $\boldsymbol{\delta}$  is benchmark exposure sensitivity<sup>13</sup>. We would like to know what impact inclusion of  $Z$  has on the model, including but not limited to its impact on returns  $\mathbf{y}$ <sup>14</sup>. For example, if inclusion of  $Z$  has no impact, then  $\boldsymbol{\gamma}$  is statistically zero: our portfolio manager's choice of  $Z$  is not generating *alpha*. In the sequel our analyses are based on the following

**Assumption 2.1** (Filtered probability space).  $(\Omega, \mathcal{F}, \mathbb{F}, P)$ .  $\Omega$  is the sample space for states of nature;  $\mathcal{F}$  is the  $\sigma$ -field of Borel measurable subsets of  $\Omega$ ;  $P$  is a probability measure defined on  $\Omega$ ; and  $\mathbb{F} = \{\mathcal{F}_s \subseteq \mathcal{F}; 0 \leq s < t < \infty\}$  is a filtration of sub  $\sigma$ -fields of  $\mathcal{F}$ .

**Assumption 2.2.**  $y : \Omega \rightarrow \mathbb{R} \setminus \{-\infty, \{\infty\}\}$

**Assumption 2.3.**  $P - \lim_{T \rightarrow \infty} \frac{\sum_{t=1}^T x_t z_t}{T} = 0$

**Assumption 2.4.** The hedge factor matrix  $Z(t, \omega) = (z_{ij}(t, \omega)) \in L^2(\Omega, \mathcal{F}, P)$ . Thus

i.  $(z_{ij}(t, \omega))$  is  $\mathcal{B}[0, \infty) \otimes \mathcal{F}$  measurable for the  $\sigma$ -field of Borel sets  $\mathcal{B}$  generated on  $[0, \infty)$ .

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<sup>7</sup>See e.g., [Urstadt \(2010\)](#).

<sup>8</sup>See e.g. [Treynor and Mazuy \(1966\)](#); [Treynor and Black \(1973\)](#); [Merton \(1981\)](#); [Bollen and Busse \(2001\)](#). Cf. [Grinblatt and Titman \(1989\)](#); [Ferson and Schadt \(1996\)](#).

<sup>9</sup>See [Grinold and Kahn, 2000](#), pp. 88-89) for explanation of benchmarking concept.

<sup>10</sup>Arguably the most popular augmented CAPM-type benchmarking model is [Fama and French \(1993\)](#) (3-factor model includes; benchmark; small minus big stock returns (SMB); high minus low book to market stock returns (HML)). See [Noehel et al. \(2010\)](#) for a literature review.

<sup>11</sup>Column vectors are in bold print. The superscript T corresponds to transposition of a vector or matrix accordingly.

<sup>12</sup>[Jarrow and Protter, 2010](#), pg. 2) identifies the constant intercept in a multifactor model as portfolio alpha. Our approach is tantamount to explaining that intercept with  $Z$ . See [Avery et al., 2011](#), pg. 17-18).

<sup>13</sup>See e.g. [Treynor and Black, 1973](#), pg. 68) for further interpretation and analytics.

<sup>14</sup>[MacKinlay and Pastor, 1998](#), pg. 5) posited a similar parametrization except that they used a [James and Stein \(1961\)](#) type estimation procedure to evaluate the impact of a missing factor on returns.

ii.  $z_{ij}(t, \omega)$  is  $\mathcal{F}_t$ -adapted.

iii.  $E[z_{ij}^2(t, \omega)] < \infty$ .

**Assumption 2.5.** Markets are liquid so trades are executed at given prices.

**Assumption 2.6.** Market microstructure effects are negligible.

**Assumption 2.7.**  $E[\varepsilon] = 0$ ,  $E[\varepsilon^2] < \infty$

**Assumption 2.8.**  $\beta$  is time varying.

To facilitate our asymptotic theory of portfolio alpha, we use a canonical dyadic partition of the unit interval  $[0, 1]$  starting at an arbitrary time  $t = t_0$ , on function space  $C[0, 1]$ <sup>15</sup>. In particular,  $\Pi^{(n)} = \{t_0^{(n)}, t_1^{(n)}, \dots, t_{m_n}^{(n)}\}$  is a dyadic partition  $t_j^{(n)} = j \cdot 2^{-n}$  for  $j = 1 \dots 2^n$ . Let  $y_{t_j^{(n)}}$  be the augmented portfolio return at time  $t_j^{(n)}$ ;  $\mathbf{x}_{t_j^{(n)}}^T$  be the corresponding row vector of returns on the benchmark assets; and  $\mathbf{z}_{t_j^{(n)}}^T$  be the corresponding row vector of returns on hedge factors in the model. Let  $\delta_{t_j^{(n)}}$  and  $\gamma_{t_j^{(n)}}$  be the  $t_j^{(n)}$ -th period coefficients, and

$$\Delta \delta_{t_{j+1}^{(n)}} = \delta_{t_{j+1}^{(n)}} - \delta_{t_j^{(n)}}, \quad \Delta \gamma_{t_{j+1}^{(n)}} = \gamma_{t_{j+1}^{(n)}} - \gamma_{t_j^{(n)}} \quad (2.3)$$

be the corresponding change in model coefficients due to an additional observation<sup>16</sup>.

**Assumption 2.9.**  $\Delta \delta_{t_{j+1}^{(n)}}$  and  $\Delta \gamma_{t_{j+1}^{(n)}}$  are  $\mathcal{F}_{t_{j+1}^{(n)}}$ -measurable.

To isolate the impact of the  $j + 1$ -th period observation on the model we write

$$\begin{pmatrix} y_{t_j^{(n)}} \\ \dots \\ y_{t_{j+1}^{(n)}} \end{pmatrix} = \begin{pmatrix} X_{t_j^{(n)}} & \vdots & Z_{t_j^{(n)}} \\ \dots & \dots & \dots \\ \mathbf{x}_{t_{j+1}^{(n)}}^T & \vdots & \mathbf{z}_{t_{j+1}^{(n)}}^T \end{pmatrix} \begin{pmatrix} \hat{\delta}_{t_{j+1}^{(n)}} \\ \dots \\ \hat{\gamma}_{t_{j+1}^{(n)}} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_{t_j^{(n)}} \\ \dots \\ e_{t_{j+1}^{(n)}} \end{pmatrix} \quad (2.4)$$

where  $e$  is the sample estimate of  $\varepsilon$ . In which case we get the linear relation

$$\mathbf{y}_{t_j^{(n)}} = X \hat{\delta}_{t_{j+1}^{(n)}} + Z \hat{\gamma}_{t_{j+1}^{(n)}} + \mathbf{e}_{t_{j+1}^{(n)}} \quad (2.5)$$

$$y_{t_{j+1}^{(n)}} = \mathbf{x}_{t_{j+1}^{(n)}}^T \hat{\delta}_{t_{j+1}^{(n)}} + \mathbf{z}_{t_{j+1}^{(n)}}^T \hat{\gamma}_{t_{j+1}^{(n)}} + e_{t_{j+1}^{(n)}} \quad (2.6)$$

where  $\mathbf{x}_{t_{j+1}^{(n)}}^T$  and  $\mathbf{z}_{t_{j+1}^{(n)}}^T$  are row vectors. So that if there are  $m$  assets in the benchmark portfolio, and  $p$  hedge factors/assets, then  $X_{t_j^{(n)}} = [\mathbf{x}_{t_1^{(n)}} \dots \mathbf{x}_{t_j^{(n)}}]$  is a  $j \times m$  matrix, and  $Z_{t_j^{(n)}} = [\mathbf{z}_{t_1^{(n)}} \dots \mathbf{z}_{t_j^{(n)}}]$  is a  $j \times p$  matrix. An additional observation appends a row vector to each matrix accordingly<sup>17</sup>. So that  $Z$  is really a progressively measurable  $j \times p$  matrix process for  $j = 0, 1, \dots, 2^n$ .

<sup>15</sup>Technical points involving Skorokhod space  $D[0, 1]$  are ignored here.

<sup>16</sup>(Fulkerson et al., 2010, pp. 8-9) used a similar parametrization to decompose portfolio returns into active and passive components.

<sup>17</sup>In the sequel we suppress the time subscript for the  $X_{t_j^{(n)}}$  and  $Z_{t_j^{(n)}}$  matrices, and write  $X$  and  $Z$  for notational convenience. However, we reserve the right to invoke the time subscript as necessary..

## 2.1 Behavioural Heuristics On Altering Beta

Technically,  $\mathbf{z}_{t_{j+1}}^T(\omega)$  is not  $\mathcal{F}_{t_j}^{(n)}$ -adapted. That is, it cannot be determined solely from information in  $\mathcal{F}_{t_j}^{(n)}$ . The portfolio manager must be “clairvoyant” and find some algebraic number<sup>18</sup> in  $\mathcal{F}_{t_{j+1}}^{(n)}$ . The gist of Cadogan (2011b) is that implied volatility ( $\sigma$ ) from options prices is such a “clairvoyant” algebraic number<sup>19</sup>. Therefore, for some closed class of polynomials  $\mathcal{P}$ , and polynomials  $g, h \in \mathcal{P}$ , the hedge factor(s)  $\mathbf{z}_{t_{j+1}}^T$  can be expressed as a polynomial  $g(\sigma)$  for  $\sigma \in \mathcal{F}_{t_{j+1}}^{(n)}$  with coefficients drawn from  $\mathcal{F}_{t_j}^{(n)}$ . In other words, returns forecast must be based on forward  $[g(\sigma)]$  and backward  $[h(y_{t_1}^{(n)}, y_{t_2}^{(n)}, \dots, y_{t_j}^{(n)})]$  looking variables based on derivative pricing. So that

$$y_{t_{j+1}}^{(n)} = g(\sigma) + h(y_{t_1}^{(n)}, y_{t_2}^{(n)}, \dots, y_{t_j}^{(n)}) + \varepsilon_{t_{j+1}}^{(n)} \quad (2.7)$$

In which case for  $\mathbf{x}_{t_j}^{(n)}$  fixed in 2.6,  $\mathbf{z}_{t_{j+1}}^{(n)} = g(\sigma)$  is the contribution of new information to returns,  $y_{t_{j+1}}^{(n)}$ , after parameter updates<sup>20</sup>. In a nutshell,  $\mathbf{z}_{t_{j+1}}^{(n)}$  is predictable<sup>21</sup>; thus paving the way for its use in martingale transform equations. These results are summarized in the following

**Lemma 2.10** (Predictable hedge factors).

Let  $\mathbf{z}_{t_{j+1}}^T$  be a vector of returns isomorphic to the terminal payoff of a contingent claim, and  $\sigma$  be an algebraic number in  $\mathcal{F}_{t_{j+1}}^{(n)}$ . Let  $\mathcal{P}$  be the class of closed polynomials with coefficients in  $\mathcal{F}_{t_j}^{(n)}$ . Then  $\mathbf{z}_{t_{j+1}}^T = g(\sigma)$  is predictable.

□

*Remark 2.1.* Kassouf (1969) provides empirical support for this lemma.

The dispositive question here is how to alter the portfolio’s beta, i.e., forecast  $\delta_{t_{j+1}}^{(n)}$  and  $\gamma_{t_{j+1}}^{(n)}$ , to maximize next period’s returns. The vector of returns is given by

$$\mathbf{y}_{t_{j+1}}^{(n)} = [\mathbf{y}_{t_j}^T : y_{t_{j+1}}^T]^T. \text{ Whereupon} \quad (2.8)$$

$$\mathbf{y}_{t_j}^{(n)} = X\delta_{t_{j+1}}^{(n)} + Z\gamma_{t_{j+1}}^{(n)} + \varepsilon_{t_j}^{(n)} \quad (2.9)$$

$$= X\delta_{t_j}^{(n)} + \underbrace{X\Delta\delta_{t_{j+1}}^{(n)} + Z\gamma_{t_{j+1}}^{(n)} + \varepsilon_{t_j}^{(n)}}_{\text{ex post tracking error}}, \text{ and} \quad (2.10)$$

$$y_{t_{j+1}}^{(n)} = \mathbf{x}_{t_{j+1}}^T \delta_{t_{j+1}}^{(n)} + \underbrace{\mathbf{z}_{t_{j+1}}^T \gamma_{t_{j+1}}^{(n)}}_{\text{ex ante tracking error}} + \varepsilon_{t_{j+1}}^{(n)} \quad (2.11)$$

Ideally, the portfolio manager would like tracking error to be zero as she tries to replicate the benchmark and or index in 2.10. See e.g., (Elton et al., 2003, pp. 676-677). See also, (Grinold and Kahn, 2000, pg. 49) who define “tracking error” as “how well the portfolio can track the benchmark”. It is the “active returns”

<sup>18</sup>See e.g., (Clark, 1971, pg. 88) for definition of algebraic number and related concepts introduced here.

<sup>19</sup>See also, Bakshi et al. (2010) who showed that forward looking volatility, i.e. an algebraic number, from options market have predictive power for asset returns. At a more technical level, (Myneni, 1992, pg. 10) used martingale theory from (Dellacherie and Meyer, 1982, pg. 135, 74(b)) to advocate for the existence of *dual predictable projection* of processes with integrable variation. To wit, if  $Z$  is convex—as hypothesized, then it satisfies the dual predictable projection criterion.

<sup>20</sup>See e.g., Admati and Pfleiderer (1988) for evolution of trade patterns and information flows.

<sup>21</sup>See (Karatzas and Shreve, 1991, pg. 21).

on the portfolio. This is tantamount to imposing the following behavioral restrictions on the *ex post* tracking error equation

$$X\Delta\boldsymbol{\delta}_{t_{j+1}}^{(n)} + Z\boldsymbol{\gamma}_{t_{j+1}}^{(n)} + \boldsymbol{\varepsilon}_{t_j}^{(n)} = 0 \quad (2.12)$$

If the proportion of assets in the benchmark is fixed—technically this is a "portable alpha" strategy, then

$$\Delta\boldsymbol{\delta}_{t_{j+1}}^{(n)} = 0, \text{ and} \quad (2.13)$$

$$\hat{\boldsymbol{\gamma}}_{t_{j+1}}^{\text{res}} = -(Z^T Z)^{-1} Z^T \boldsymbol{\varepsilon}_{t_j}^{(n)} \quad (2.14)$$

Thus, hedge factor exposure sensitivity plainly depends on, *inter alia*, the behavior of  $\boldsymbol{\varepsilon}_{t_j}^{(n)}$ . Consistent with our augmented model, define the projection matrices, see e.g., (Greene, 2003, pp. 149-150)

$$P_X = X(X^T X)^{-1} X^T \quad (2.15)$$

$$P_Z = Z(Z^T Z)^{-1} Z^T \quad (2.16)$$

$$M_X = I - P_X \quad (2.17)$$

$$M_Z = I - P_Z \quad (2.18)$$

So that assuming that  $X$  and  $Z$  are uncorrelated with  $\boldsymbol{\varepsilon}$  we have the unrestricted estimate, see (Christopherson et al., 1998, pp. 121-122),

$$\hat{\boldsymbol{\gamma}}_{t_{j+1}}^{\text{unres}} = (Z^T M_X Z)^{-1} Z^T M_X \mathbf{y}_{t_j}^{(n)} \quad (2.19)$$

Our portfolio manager has superior market timing ability, see (Ferson and Schadt, 1996, pg. 436), if

$$\mathbf{z}_{t_{j+1}}^T \boldsymbol{\gamma}_{t_{j+1}}^{\text{res}} + \boldsymbol{\varepsilon}_{t_{j+1}}^{(n)} \geq 0 \quad (2.20)$$

So we can rewrite 2.11 as follows

$$y_{t_{j+1}}^{(n)} = \mathbf{x}_{t_{j+1}}^T \boldsymbol{\delta}_{t_{j+1}}^{(n)} + \max\{0, \mathbf{z}_{t_{j+1}}^T \boldsymbol{\gamma}_{t_{j+1}}^{(n)} + \boldsymbol{\varepsilon}_{t_{j+1}}^{(n)}\} \quad (2.21)$$

Whereupon substitution of  $\hat{\boldsymbol{\gamma}}_{t_{j+1}}^{\text{res}}$  from 2.14 in 2.20 yields

$$\boldsymbol{\varepsilon}_{t_{j+1}}^{(n)} \geq \mathbf{z}_{t_{j+1}}^T (Z^T Z)^{-1} Z^T \boldsymbol{\varepsilon}_{t_j}^{(n)} \quad (2.22)$$

The functional form in 2.21 is equivalent to (Merton, 1981, pp. 365-366, 368-369) formulation of isomorphism between the pattern of returns from market timing and returns on an option strategy<sup>22</sup>. Intuitively, our parametrization implies that the benchmark is perfectly tracked. Thus, any mispricing in the model stems from the PM performance in selecting hedge factors or contingent claims. In any event, 2.22 suggests that if our portfolio manager is bullish, i.e. she believes that the returns process is a semi-martingale that is favorable to her, see e.g., (Doob, 1953, pg. 299), then

$$\mathbf{z}_{t_{j+1}}^T (Z^T Z)^{-1} Z^T \boldsymbol{\varepsilon}_{t_j}^{(n)} \geq \boldsymbol{\varepsilon}_{t_j}^{(n)} \quad (2.23)$$

Equations 2.22 and 2.23 gives rise to the following

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<sup>22</sup>See also, Glosten and Jagannathan (1994) and (Agarwal and Naik, 2004, pg. 68) for extension(s).

**Theorem 2.11** (Market Timing Theorem). *Let  $Z$  be a matrix of hedge factors at time  $t_j^{(n)}$  and  $\mathbf{z}_{t_{j+1}^{(n)}}^T$  be an additional row vector of future observations, i.e., derivative prices of the hedge factors. Furthermore, suppose that  $\boldsymbol{\varepsilon}_{t_j^{(n)}}$  is a vector of portfolio manager forecast errors, and  $\varepsilon_{t_{j+1}^{(n)}}$  is forecast error at time  $t_{j+1}^{(n)}$ . Assume that  $Z$  and  $\boldsymbol{\varepsilon}$  are uncorrelated, and that  $\boldsymbol{\varepsilon} \sim (0, 1)$ . Then our portfolio manager has market timing ability iff*

$$\sup_{0 \leq j \leq 2^n} \|E[\mathbf{z}_{t_{j+1}^{(n)}}^T](Z^T Z)^{-1} Z^T\|^2 \geq 2^{-n} \quad (2.24)$$

□

*Remark 2.2.* The theorem essentially implies that as trading frequency increases, i.e.  $n \uparrow \infty$ , our portfolio manager will have timing ability for any previsible process  $\{\mathbf{z}_t, \mathcal{F}_t; t \geq 0\}$ . This is the *sui generis* of market timing. It constitutes a mathematical proof of [Chance and Hemler \(2001\)](#) empirical results which found that the same portfolio managers who seemingly lacked timing ability at low frequency were found to have timing ability at high frequency.

## 2.2 The Martingale System Equation For Market Timing

This section develops the martingale representation theory. See ([Dudley, 2004](#), pp. 363-365) and ([Breiman, 1968](#), Chapter 5) for excellent summary of martingales. Let

$$u_j(\omega) = \begin{cases} 1 & \text{if } \mathbf{z}_{t_{j+1}^{(n)}}^T \boldsymbol{\gamma}_{t_{j+1}^{(n)}}^{\text{res}} + \varepsilon_{t_{j+1}^{(n)}}(\omega) > 0 \\ 0 & \text{if } \mathbf{z}_{t_{j+1}^{(n)}}^T \boldsymbol{\gamma}_{t_{j+1}^{(n)}}^{\text{res}} + \varepsilon_{t_{j+1}^{(n)}}(\omega) \leq 0 \end{cases} \quad (2.25)$$

and define

$$d_{k+1} = y_{t_{k+1}^{(n)}} - \mathbf{x}_{t_k^{(n)}}^T \boldsymbol{\delta}_{t_{k+1}^{(n)}} \quad (2.26)$$

So that the equation

$$\bar{d}_n = d_1 + \sum_{j=1}^{n-1} u_j(\omega) d_{j+1} \quad (2.27)$$

represents the excess returns from the given portfolio strategy. This is the martingale system equation referred to in ([Snell, 1952](#), pg. 295). In the context of our model it represents the portfolio manager data mining algorithm which propels her high frequency trades. The specific strategy in place can be seen from rewriting the equation as

$$\bar{d}_n = d_1 + \sum_{k=1}^{2^n-1} (y_{t_{k+1}^{(n)}} - \mathbf{x}_{t_k^{(n)}}^T \boldsymbol{\delta}_{t_{k+1}^{(n)}})^+ \quad (2.28)$$

$$= d_1 + \sum_{k=1}^{2^n-1} (\mathbf{z}_{t_{k+1}^{(n)}}^T \boldsymbol{\gamma}_{t_{k+1}^{(n)}} + \varepsilon_{t_{k+1}^{(n)}})^+ \quad (2.29)$$

where the summand is tantamount to a call option on the benchmark<sup>23</sup>, as indicated by [Merton \(1981\)](#); [Henriksson and Merton \(1981\)](#). See also, ([Henriksson, 1984](#), pg. 77). According to ([Snell, 1952](#), Thm. 2.1, pg. 295)

<sup>23</sup>In our case, the call option is on some hedge factor(s) that are uncorrelated with the benchmark *per se*. Arguably, the benchmark constitutes the "microforecast" while the hedge factor(s) comprise the "macroforecast" or market timing ability. See [Fama \(1972\)](#).

the sequence  $\{d_n, \mathcal{F}_n; n \geq 1\}$  is a semimartingale in which  $E[\bar{d}_n | \mathcal{F}_1] \leq E[d_n | \mathcal{F}_1]$ . For our purposes it implies that in an efficient market, in the long run, the portfolio manager should be no better off by “judicious” selection of favorable  $\bar{d}_n$  transforms, i.e., option(s) strategies. These artifacts are summarized in a slightly modified version of Snell’s Theorem as follows:

**Proposition 2.12** (Snell’s Theorem). ([Snell, 1952](#), Thm. 2.1, pg. 295).

Let  $(\Omega, \mathcal{P}, \mathcal{F})$  be a probability space;  $D = \{d_k, \mathcal{F}_k; k \geq 1\}$  be a martingale; and  $\{u_k(\omega); k \geq 1\}$  be a sequence of  $\mathcal{F}_k$ -measurable random variables. Define

$$\bar{d}_k = d_1 + \sum_{j=1}^{k-1} u_j(\omega) d_{k+1}$$

If  $E[|\bar{d}_k|] < \infty$  for all  $k$ , then  $\bar{D} = \{\bar{d}_k, \mathcal{F}_k; k \geq 1\}$  is a martingale, and the  $u_k$ ’s are nonnegative, then  $\bar{D}$  is a submartingale. If the  $u_k$ ’s are binary random variables taking the values 0 or 1, then we have

$$E[\bar{d}_k | \mathcal{F}_1] \leq E[d_k | \mathcal{F}_1]$$

with probability 1. □

*Proof.* See [Snell \(1952\)](#). □

### 2.3 Trade strategy in continuous time, and statistical test for market timing

In this subsection we state some of our main results—most with referenced proofs. Equating [2.14](#) and [2.19](#) gives rise to the following

**Proposition 2.13** (Spectral test for market timing).

Let  $Z$  be a  $j \times p$  matrix of hedge factors,  $X$  be a  $j \times m$  matrix of benchmark assets, and  $P_X = X(X^T X)^{-1} X^T$  be the projection matrix on  $X$ -space. Define  $A = Z^T (2I - P_X) Z$  where  $I$  is the identity matrix. Let  $\lambda_k(A)$  be the  $k$ -th eigenvalue of  $A$ . Let  $\eta > 0$  be a suitably chosen number. Then our portfolio manager has timing ability if

$$\max_{1 \leq k \leq p} |\lambda_k(A)| > \eta$$

Moreover, this is tantamount to the statistical test:

$$H_0 : \max_{1 \leq k \leq p} \lambda_k(A) \leq \eta \quad \text{versus} \quad H_a : H_0 \text{ is not true}$$

□

*Remark 2.3.* The exact statistical distribution for  $\max_{1 \leq k \leq p} \lambda_k(A)$  is a fairly complex looking function given in [Erten et al. \(2009\)](#). Moreover, in practice it is possible for  $\lambda$  to be negative based on numerical routines.

*Remark 2.4.* ([Hansen and Scheinkman, 2009](#), Cor. 6.1, pg. 200) derived a principal eigenvalue result by applying semigroup theory to a stochastic discount factor assumed to follow a Markov process.

Nonetheless, to computer the power of our spectral test we proffer the following

**Theorem 2.14** (Power of spectral test for market timing). If  $\ell_1 = \max_{1 \leq k \leq p} \lambda_k(A)$  is the largest latent root of  $A$ , and  $A = H^T H$ , where  $H \sim N(0, I_n \otimes \Sigma)$  and  $W_p(n, \Sigma)$  is a Wishart distribution with  $n$ -degrees of freedom and dimension  $p$ ,  $A \sim W_p(n, \Sigma)$ , then the distribution function for  $\ell_1$  can be expressed as

$$P_\Sigma(\ell < \eta) = \frac{\Gamma_m[\frac{1}{2}(m+1)]}{\Gamma_m[\frac{1}{2}(n+m+1)]} \det(\frac{1}{2}n\eta\Sigma^{-1})^{\frac{n}{2}} {}_1F_1(\frac{n}{2}; \frac{1}{2}(n+m+1); -\frac{1}{2}n\eta\Sigma^{-1}) \quad (2.30)$$



where  ${}_1F_1(\cdot)$  is a hypergeometric function such that

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}$$

where  $(a)_k = a(a+1) \dots (a+k-1)$ , and  $\Gamma_m(\cdot)$  is a multivariate gamma function.

*Proof.* See (Muirhead, 2005, pg. 421, Cor. 9.7.2).  $\square$

**Remark 2.5.** The multivariate gamma function  $\Gamma_m(\cdot)$  is defined in (Muirhead, 2005, pg. 61).

**Theorem 2.15** (Subordinated Brownian motion). *Let  $\varepsilon_{t_j^{(n)}}$  be independent and identically distributed with  $E[\varepsilon_{t_j^{(n)}}] = 0$  and  $E[\varepsilon_{t_j^{(n)}}^2] = \sigma^2 < \infty$ , for  $j = 1, 2, \dots, 2^n$ . Let  $S_N = \sum_{j=1}^N \varepsilon_{t_j^{(n)}}$  and for  $t_j^{(n)} \leq t < t_{j+1}^{(n)}$  define*

$$\varepsilon_t^{(n)} = \frac{1}{\sqrt{n}} [S_{[nt]} + (nt - [nt])\varepsilon_{[nt]+1}^{(n)}]$$

*Then  $\varepsilon_{t+2^{-n}}^{(n)} - \varepsilon_t^{(n)}$  is a subordinated Brownian motion for some strictly monotone function  $c(\cdot)$ . In particular,  $\varepsilon_{t+2^{-n}}^{(n)}(\omega) - \varepsilon_t^{(n)}(\omega) \sim B_{c(t)}(\omega)$  on the probability space  $(\Omega, \mathcal{F}, P)$ .*

*Proof.* See Appendix A.  $\square$

**Theorem 2.16** (Trading strategy representation. Cadogan (2011a)).

*Let  $(\Omega, \mathcal{F}, \mathbb{F}, P)$  be a filtered probability space, and  $Z = \{Z_s, \mathcal{F}_s; 0 \leq s < \infty\}$  be a hedge factor matrix process on the augmented filtration  $\mathbb{F}$ . Furthermore, let  $a^{(i,k)}(Z_s)$  be the  $(i,k)$ -th element in the expansion of the transformation matrix  $(Z_s^T Z_s)^{-1} Z_s^T$ , and  $B = \{B(s), \mathcal{F}_s; s \geq 0\}$  be Brownian motion adapted to  $\mathbb{F}$  such that  $B(0) = x$ . Let  $\gamma^{(i)\Pi^{(n)}}(t, \omega) = -\sum_{k=1}^j a^{(i,k)}(Z_{t_k^*}^{(n)}) \varepsilon_{t_k^{(n)}} \chi_{[t_{j-1}^{(n)}, t_j^{(n)}]}(t)$ ,  $t_{j-1}^{(n)} < t_k^* < t_j^{(n)}$ , with respect to partition  $\Pi^{(n)}$  and characteristic function  $\chi_{[t_{j-1}^{(n)}, t_j^{(n)}]}(t)$ . Assuming that  $B$  is the background driving Brownian motion for high frequency trading, the limiting hedge factor sensitivity process, i.e. trading strategy,  $\gamma = \{\gamma_s, \mathcal{F}_s; 0 \leq s < \infty\}$  generated by portfolio manager market timing for Brownian motion starting at the point  $x \geq 0$  has representation*

$$d\gamma^{(i)}(t, \omega) = \sum_{k=1}^j a^{(i,k)}(Z_t) \left[ \frac{x}{1-t} \right] dt - \sum_{k=1}^j a^{(i,k)}(Z_t) dB(t, \omega), \quad x \geq 0$$

*for the  $i$ -th hedge factor  $i = 1, \dots, p$ , and  $0 \leq t \leq 1$ .*

*Proof.* Apply Theorem 2.15 to  $\lim_{n \rightarrow \infty} \gamma^{(i)\Pi^{(n)}}(t, \omega)$ . See (Cadogan, 2011a, Thm. 4.6).  $\square$

### 3. Application: Dynamic alpha in a single factor model

We employ our trade strategy representation theorem, to shed light on the behavior of portfolio *alpha* in a single factor model like CAPM, where there is no hedge factor. In particular, let  $\mathbb{1}_{\{n\}}$  be a  $n \times 1$  vector, and

$$Z = \mathbb{1}_{\{n\}} \tag{3.1}$$

So that

$$(Z^T Z)^{-1} Z^T = n^{-1} \mathbb{1}_{\{n\}}^T, \text{ and } a^{(1,k)}(Z_s) = n^{-1}, \quad k = 1, \dots, n \tag{3.2}$$



Substitution of these values in 2.2 and Theorem 2.16 gives us

$$\alpha^{(1)}(t) = \gamma^{(1)}(t) \quad (3.3)$$

$$-d\alpha^{(1)}(t) = -\frac{x}{1-t}dt + dB(t) \quad (3.4)$$

That is the equation of a Brownian bridge starting at  $B(0) = x$  on the interval  $[0, 1]$ . See (Karlin and Taylor, 1981, pg. 268). So that

$$d\alpha^{(1)}(t) = -dB^{br}(t) \quad (3.5)$$

$$\alpha^{(1)}(t) = B^{br}(0) - B^{br}(t) \quad (3.6)$$

The Brownian bridge feature suggests that portfolio managers open and close their net positions at zero, and take profits (or losses) in between. See e.g., Urstadt (2010). And the negative sign implies that our portfolio manager is engaged in a price reversal strategy. See e.g., (Brogaard, 2010, pp. 14-15). So that  $B^{br}(t) < 0 \Rightarrow \alpha^{(1)}(t) > 0$ . According to Girsanov's formula in (Øksendal, 2003, pg. 162), we have an equivalent probability measure  $Q$  based on the martingale transform

$$M(t, \omega) = \exp\left(\int_0^t \frac{x}{1-s} dB(s, \omega) - \int_0^t \left(\frac{x}{1-s}\right)^2 ds\right) \quad (3.7)$$

$$dQ(\omega) = M(T, \omega) dP(\omega), \quad 0 \leq t \leq T \leq 1 \quad (3.8)$$

Thus, we have the  $Q$ -Brownian motion, i.e. Brownian bridge

$$\hat{B}(t) = -\int_0^t \frac{x}{1-s} ds + B(t), \quad \text{and} \quad (3.9)$$

$$d\alpha^{(1)}(t) = -d\hat{B}(t) = -dB^{br}(t) \quad (3.10)$$

In other words,  $\alpha^{(1)}$  is a  $Q$ -Brownian motion, i.e. Brownian bridge, that reverts to the origin starting at  $x$ . We note that for idiosyncratic risk  $\varepsilon(t)$ , the CAPM holds if  $\alpha^{(1)}(t) + \varepsilon(t) = 0$ , and

$$\frac{d\alpha^{(1)}(t)}{dt} + \frac{\varepsilon(t)}{dt} = -\frac{dB^{br}(t)}{dt} + \frac{\varepsilon(t)}{dt} = 0 \quad (3.11)$$

Hence the "residual(s)"  $\varepsilon(t)$ , associated with alpha, have an approximately skewed U-shape pattern if  $B^{br}(t) \leq 0$ . (Karatzas and Shreve, 1991, pg. 358) also provide further analytics which show that on  $[0, 1]$  we can write the portfolio alpha process in mean reverting form as

$$d\alpha^{(1)}(t) = \frac{1 - \alpha^{(1)}(t)}{1-t} dt + dB(t); \quad 0 \leq t \leq 1, \quad \alpha^{(1)}(0) = 0 \quad (3.12)$$

$$M(t) = \int_0^t \frac{dB(s)}{1-s} \quad (3.13)$$

$$T(s) = \inf\{t | M_t > s\} \quad (3.14)$$

$$G(t) = B_{<M>T(t)} \quad (3.15)$$

Thus, under Dambis-Dubins-Schwarz criteria, alpha is a time changed martingale—in this case Brownian motion. In the absence of a hedge factor, the single factor or benchmark, is perfectly tracked if

$$\alpha^{(1)}(t) = 0, \quad \hat{B}(t, \omega) = x \quad (3.16)$$

The foregoing gives rise to the following

**Theorem 3.1** (Positive CAPM alpha excursion).

Let  $\alpha^{(1)+}(t)$  be the  $Q$ -Brownian motion excursion path of CAPM alpha at time  $t$  in 3.9, and  $B(t)$  be standard one-dimensional Brownian motion. Let  $\tau_+^\alpha(t)$  be the first zero of  $B$  after and  $\tau_-^\alpha(t)$  be the first zero of  $B$  before  $t = 1$ . So that

$$\tau_+^\alpha(t) = \inf\{t > 1 | B(t) = 0\} \quad (3.17)$$

$$\tau_-^\alpha(t) = \sup\{t < 1 | B(t) = 0\} \quad (3.18)$$

Then

$$\alpha^{(1)+}(t) = \frac{|B(t\tau_+^\alpha + (1-t)\tau_-^\alpha(t))|}{\sqrt{\tau_+^\alpha(t) - \tau_-^\alpha(t)}} \quad (3.19)$$

□

*Proof.* See [Vervaat \(1979\)](#). □

Thus, the path properties of portfolio alpha can be identified and excess returns can be computed for suitably chosen stopping times. The property  $\hat{B}(t, \omega) = x$  reduces the problem to one of local time [of a Brownian bridge] at  $x$ . We can think of  $x$  as a hurdle rate such as transaction costs that the manager must attain to break even. The probability associated with the CAPM alpha level set  $\mathfrak{B} = \{\omega | \hat{B}(t, \omega) = x\}$  is zero. However, even though that set has P-measure zero, its local time exists. Perhaps more important, the perfectly hedged portfolio problem, i.e. the CAPM problem, reduces to one of stochastic optimal control—guiding  $\alpha^{(1)}$  to a goal of 0 by keeping it as close to 0 as possible. This problem, and related ones, were solved by [Benes et al. \(1980\)](#) and in ([Karatzas and Shreve, 1991](#), Chapter 6.2).

### 3.1 On spurious econometric tests for alpha

According to ([Karlin and Taylor, 1981](#), pg. 269) the expected value of alpha starting at  $x$ , and its variance is given by

$$E[\alpha^{(1)} | \tilde{B}(t)] = -\frac{x}{1-t}, \quad \sigma_{\alpha^{(1)}}^2 = 1 \quad (3.20)$$

Let  $\{\alpha_1^{(1)}, \dots, \alpha_N^{(1)}\}$  be a sample of alphas for  $N$ -funds. Furthermore, assume that the fund alphas are pairwise correlated with correlation coefficient  $\rho_{ij}$ . Cf. ([Avery et al., 2011](#), pp. 17-19). So that

$$\alpha_i^{(1)} = \rho_{ij} \alpha_j^{(1)}, \quad |\rho_{ij}| < 1 \quad (3.21)$$

The sample mean and variance of the funds are given by

$$\bar{\alpha}_N^{(1)} = \frac{1}{N}(\alpha_1^{(1)} + \dots + \alpha_N^{(1)}) \quad (3.22)$$

$$\sigma_{\alpha^{(1)}}^2 = \frac{1}{N^2}(\sigma_{\alpha_1^{(1)}}^2 + \dots + \sigma_{\alpha_N^{(1)}}^2 + 2 \sum_{i \neq j} \text{cov}(\alpha_i^{(1)}, \alpha_j^{(1)})) \quad (3.23)$$

$$|\sum_{i \neq j} \text{cov}(\alpha_i^{(1)}, \alpha_j^{(1)})| \leq \sum_{i \neq j} |\text{cov}(\alpha_i^{(1)}, \alpha_j^{(1)})| \leq \sum_{i \neq j} |\rho_{ij}| \leq \binom{N}{2} \quad (3.24)$$

In that milieu, a  $t$ -test for the hypothesis  $H_0 : \alpha^{(1)} = 0$  has test statistic

$$t_{\bar{\alpha}_N^{(1)}} = \frac{\bar{\alpha}_N^{(1)}}{\sigma_{\bar{\alpha}_N^{(1)}}} = \frac{\bar{\alpha}_N^{(1)}}{\sqrt{\frac{1}{N} + \frac{2}{N^2} \sum_{i \neq j} \text{cov}(\alpha_i^{(1)}, \alpha_j^{(1)})}} \quad (3.25)$$

$$\geq \frac{\bar{\alpha}_N^{(1)}}{\sqrt{\frac{1}{N} + 1}}, \text{ for sufficiently large } N \quad (3.26)$$

$$\lim_{N \rightarrow \infty} t_{\bar{\alpha}_N^{(1)}} = Z_{\bar{\alpha}_\infty^{(1)}} \geq \bar{\alpha}_\infty^{(1)}, \text{ where } Z_{\bar{\alpha}_\infty^{(1)}} \text{ is a standard normal r.v.} \quad (3.27)$$

Ergodic theory<sup>24</sup> tells us that the limiting value of the test statistic  $\bar{\alpha}_\infty^{(1)}$  is a Brownian bridge<sup>25</sup>. Moreover, according to 3.20, it tends to be negative valued. Thus, an analyst could easily conclude that the sampled funds do not generate positive alpha<sup>26</sup>. Yet, we know from the path properties in Theorem 3.1 that there are stopping times for which the funds do generate positive alpha. So contrary to (Jarrow, 2010, pg. 19) false positive alpha postulate, our theory indicates that there is a false negative alpha puzzle.

## 4. Appendix

### A. Proof of subordinated Brownian motion Theorem 2.15

*Proof.* Define

$$S_N = \sum_{j=1}^N \varepsilon_{t_j^{(n)}} \quad (A.1)$$

so that

$$E[S_N^2] = \sum_{j=1}^N E[\varepsilon_{t_j^{(n)}}^2] = N\sigma^2 \quad (A.2)$$

Without loss of generality, normalize  $\varepsilon$  with  $\frac{\varepsilon}{\sigma}$  so that we have  $E[\varepsilon^2] = 1$  and

$$E\left[\left(\frac{S_N}{\sqrt{N}}\right)^2\right] = 1 \quad (A.3)$$

For  $t_j^{(n)} \leq t < t_{j+1}^{(n)}$  let

$$\varepsilon_t^{(n)} = \frac{1}{\sqrt{n}}[S_{[nt]} + (nt - [nt])\varepsilon_{[nt]+1}] \quad (A.4)$$

<sup>24</sup>See e.g., (Gikhman and Skorokhod, 1969, pg. 127)

<sup>25</sup>In this heuristic example, we ignored issues arising from seemingly unrelated regressions or confidence sets.

<sup>26</sup>(Phillips, 1998, pg. 1308) noted that the time trend component in a Brownian bridge—in our case alpha—contributes to spurious regression. Also, (Ferson et al., 2003, pg. 1398) cautioned about seemingly significant  $t$ -ratios derived from spurious regressions.

where  $[nt]$  is the integer part of  $nt$ . So that

$$\begin{aligned} \varepsilon_{t+2^{-n}}^{(n)} - \varepsilon_t^{(n)} &= \frac{1}{\sqrt{n}} [S_{[nt+n.2^{-n}]} + \\ & (nt + n.2^{-n} - [nt + n.2^{-n}])\varepsilon_{[nt+n.2^{-n}]+1}] - [S_{[nt]} + (nt - [nt])\varepsilon_{[nt]+1}] \end{aligned} \quad (\text{A.5})$$

$$\begin{aligned} &= \sum_{j=[nt]+1}^{[nt+n.2^{-n}]} \varepsilon_{t_j}^{(n)} + \\ & (nt + n.2^{-n} - [nt + n.2^{-n}])\varepsilon_{[nt+n.2^{-n}]+1} - (nt - [nt])\varepsilon_{[nt]+1} \end{aligned} \quad (\text{A.6})$$

Which implies

$$\begin{aligned} E[(\varepsilon_{t+2^{-n}}^{(n)} - \varepsilon_t^{(n)})^2 | \mathcal{F}_{t_j}^{(n)}] &= [nt + n.2^{-n}] - [nt] - 1 + \\ & (nt + n.2^{-n} - [nt + n.2^{-n}])^2 + (nt - [nt])^2 \end{aligned} \quad (\text{A.7})$$

$$= n.2^{-n} + o(1 + n^{-1}) - 1 \quad (\text{A.8})$$

This implies that

$$\begin{aligned} E[\{\frac{1}{\sqrt{n}}(\varepsilon_{t+2^{-n}}^{(n)} - \varepsilon_t^{(n)})\}^2 | \mathcal{F}_{t_j}^{(n)}] \\ = 2^{-n} + o(n^{-1} + n^{-2}) - \frac{1}{n} = c(n).2^{-n} \end{aligned} \quad (\text{A.9})$$

for some monotone increasing function  $c(\cdot)$ . See e.g., [Cadogan \(2011b\)](#).

To complete the proof of Theorem 2.15, we note that according to precepts of construction of Brownian motion, Brownian scaling, see e.g., ([Karatzas and Shreve, 1991](#), Thm. 4.17, pg. 67; and Lemma 9.4, pg. 104), and Lemma B.1 in [Cadogan \(2011a\)](#), the quantity  $\varepsilon_{t+2^{-n}}^{(n)} - \varepsilon_t^{(n)}$  is a *scaled* Brownian motion  $W(c(n)2^{-n})$  for some *monotone increasing* pre-subordinator ‘function  $0 \leq c(\cdot) \leq 1$ .  $\square$

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